Elementary Landscape Decomposition of Combinatorial Optimization Problems

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Motivation

- **Landscapes' theory is a tool for analyzing optimization problems**
- **Peter F. Stadler is one of the main supporters of the theory**

Towards a Theory of Landscapes

Peter F. Stadler

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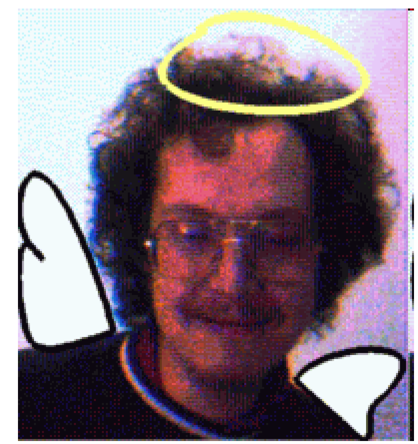


- **Applications in Chemistry, Physics, Biology and Combinatorial Optimization**
- **Central idea: study the search space to obtain information**
  - **Better understanding of the problem**
  - **Predict algorithmic performance**
  - **Improve search algorithms**

Landscape Definition

- A landscape is a triple (X,N,f) where
  - X is the solution space
  - N is the neighbourhood operator
  - f is the objective function

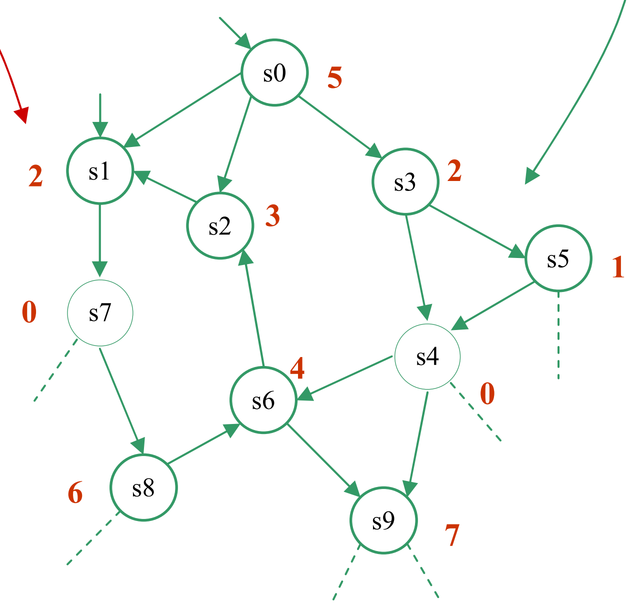
The pair (X,N) is called configuration space

- The neighbourhood operator is a function

  $N: X \rightarrow \mathcal{P}(X)$

- Solution y is neighbour of x if $y \in N(x)$
- Regular and symmetric neighbourhoods
  - $d=|N(x)| \quad \forall x \in X$
  - $y \in N(x) \Leftrightarrow x \in N(y)$
- Objective function

  $f: X \rightarrow R$ (or N, Z, Q)



Elementary Landscapes: Formal Definition

- An elementary function is an eigenvector of the graph Laplacian (plus constant)

Adjacency matrix

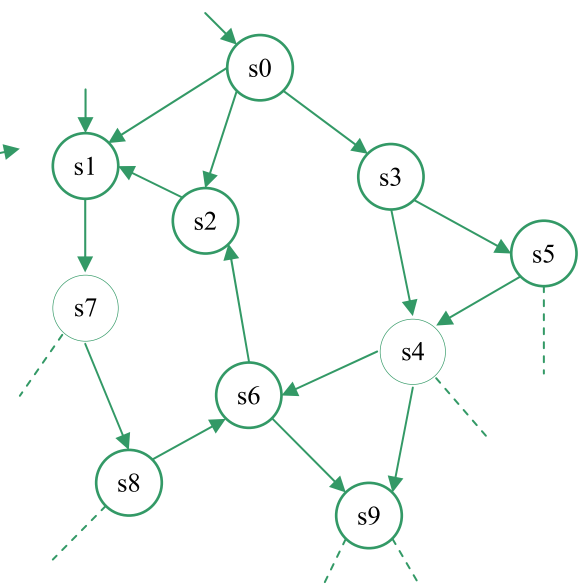
$$A_{xy} = \begin{cases} 1 & \text{if } y \in N(x) \\ 0 & \text{otherwise} \end{cases}$$

Degree matrix

$$D_{xy} = \begin{cases} |N(x)| & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$$

- Graph Laplacian: $\Delta = A - D$

Depends on the configuration space



- Elementary function: eigenvector of Δ (plus constant)

$$\Delta(f - b) = \lambda(f - b)$$

Eigenvalue

Elementary Landscapes: Characterizations

- **An elementary landscape is a landscape for which**

$$\underset{y \in N(x)}{\text{avg}}\{f(y)\} = \alpha f(x) + \beta \quad \forall x \in X$$

Depend on the problem/instance (α, β)

where

$$\underset{y \in N(x)}{\text{avg}}\{f(y)\} = \frac{1}{d} \sum_{y \in N(x)} f(y)$$

Linear relationship

- **Grover's wave equation**

$$\underset{y \in N(x)}{\text{avg}}\{f(y)\} = f(x) + \frac{k}{d}\left(\bar{f} - f(x)\right) \quad \forall x \in X$$

$$\bar{f} = \frac{1}{|X|} \sum_{y \in X} f(y)$$

Characteristic constant: *k*= - *λ*

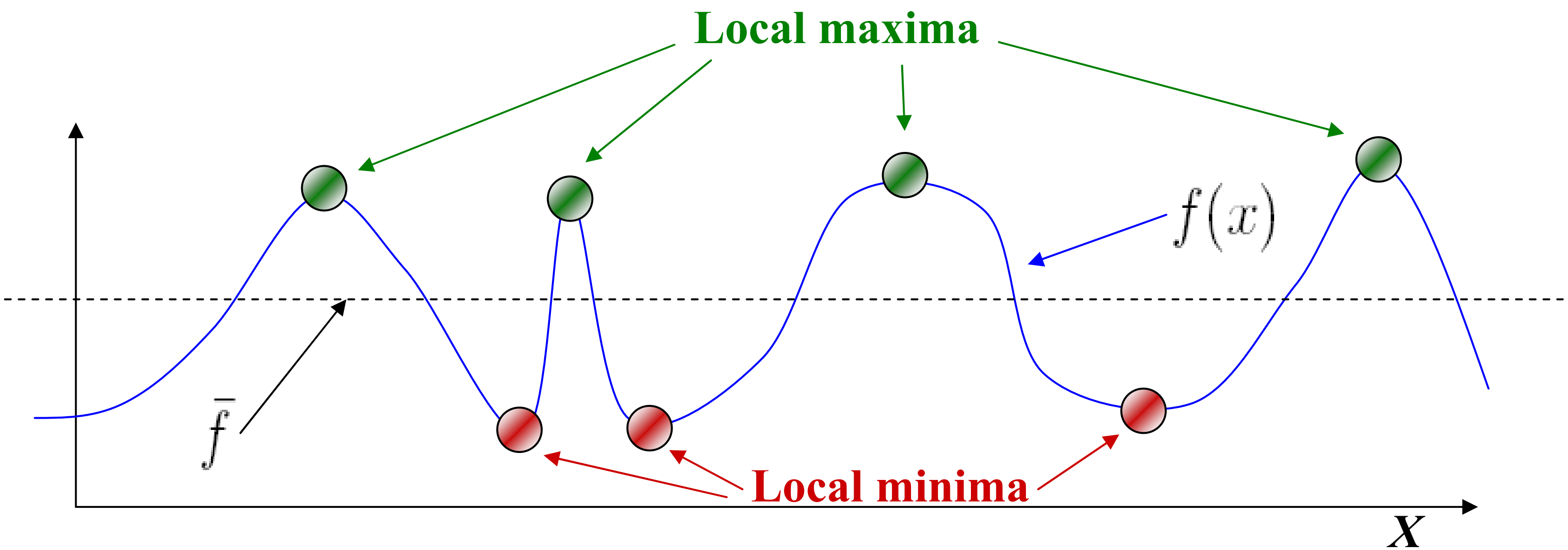
Elementary Landscapes: Properties

- **Some properties of elementary landscapes are the following**

$$f(x) < \min\left\{\operatorname*{avg}_{y \in N(x)}\{f(y)\}, \bar{f}\right\} \quad \text{or} \quad f(x) > \max\left\{\operatorname*{avg}_{y \in N(x)}\{f(y)\}, \bar{f}\right\}$$

where $f(x) \neq \bar{f}$

- **Local maxima and minima**

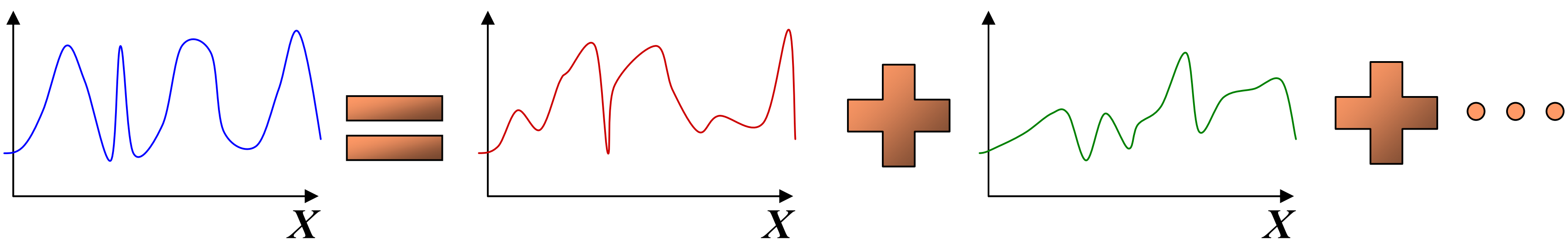


Elementary Landscapes: Examples

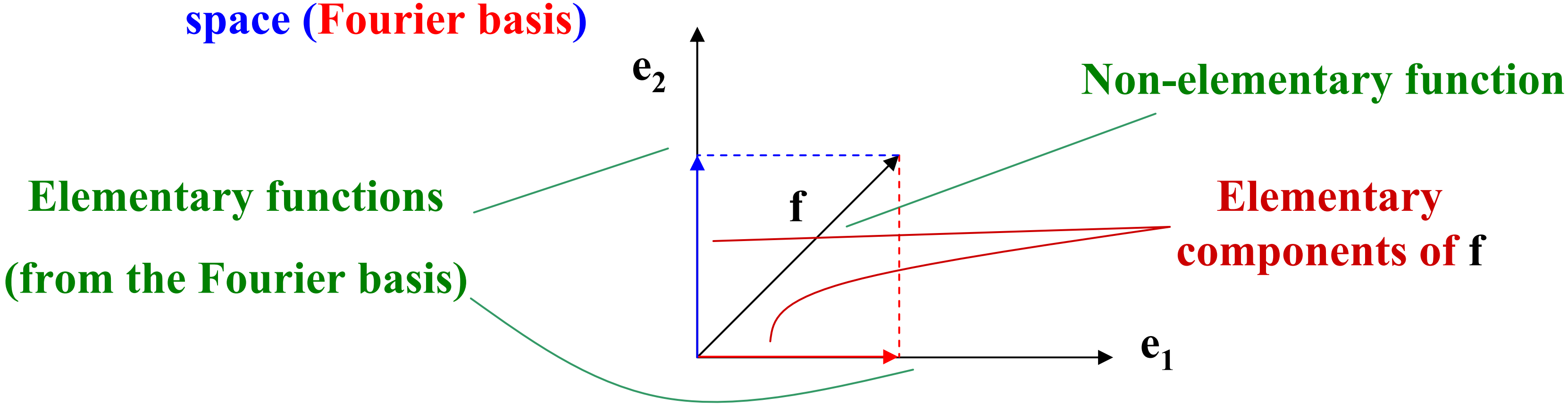
Problem	Neighbourhood	d	k
Symmetric TSP	2-opt	n(n-3)/2	n-1
	swap two cities	n(n-1)/2	2(n-1)
Antisymmetric TSP	inversions	n(n-1)/2	n(n+1)/2
	swap two cities	n(n-1)/2	2n
Graph α-Coloring	recolor 1 vertex	(α-1)n	2α
Graph Matching	swap two elements	n(n-1)/2	2(n-1)
Graph Bipartitioning	Johnson graph	$n^2/4$	2(n-1)
NEAS	bit-flip	n	4
Max Cut	bit-flip	n	4
Weight Partition	bit-flip	n	4

Landscape Decomposition

- What if the landscape is not elementary?
- Any landscape can be written as the sum of elementary landscapes



- There exists a set of eigenfunctions of Δ that form a basis of the function space (Fourier basis)



Landscape Decomposition: Walsh Functions

- If X is the set of binary strings of length n and N is the bit-flip neighbourhood then a Fourier basis is

$$\{\psi_w\} \quad w \in \{0,1\}^n$$

where

$$\psi_w(x) = (-1)^{bc(w \wedge x)} = \prod_{i=1}^{n} (-1)^{x_i w_i}$$

Bitwise AND

Bit count function (number of ones)

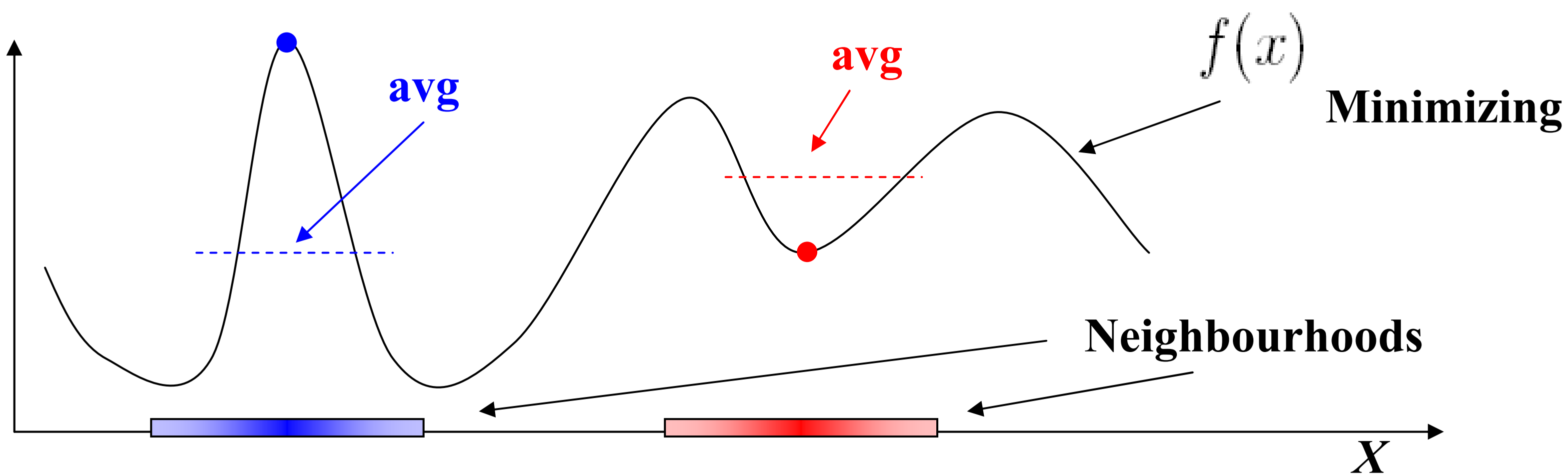
- These functions are known as Walsh Functions
- The function with subindex w is elementary with k=2 bc (w)
- In general, decomposing a landscape is not a trivial task → methodology required

Landscape Decomposition: Examples

Problem	Neighbourhood	d	Components
General TSP	inversions	n(n-1)/2	2
	swap two cities	n(n-1)/2	2
QAP	swap two elements	n(n-1)/2	3
Frequency Assignment	change 1 frequency	(α-1)n	2
Subset Sum Problem	bit-flip	n	2
MAX k-SAT	bit-flip	n	k
NK-landscapes	bit-flip	n	k+1
Radio Network Design	bit-flip	n	max. nb. of reachable antennae

New Selection Strategy

- **Selection operators usually take into account the fitness value of the individuals**



- **We can improve the selection operator by selecting the individuals according to the average value in their neighbourhoods**

New Selection Strategy

- **In elementary landscapes the traditional and the new operator are the same!**

Recall that...

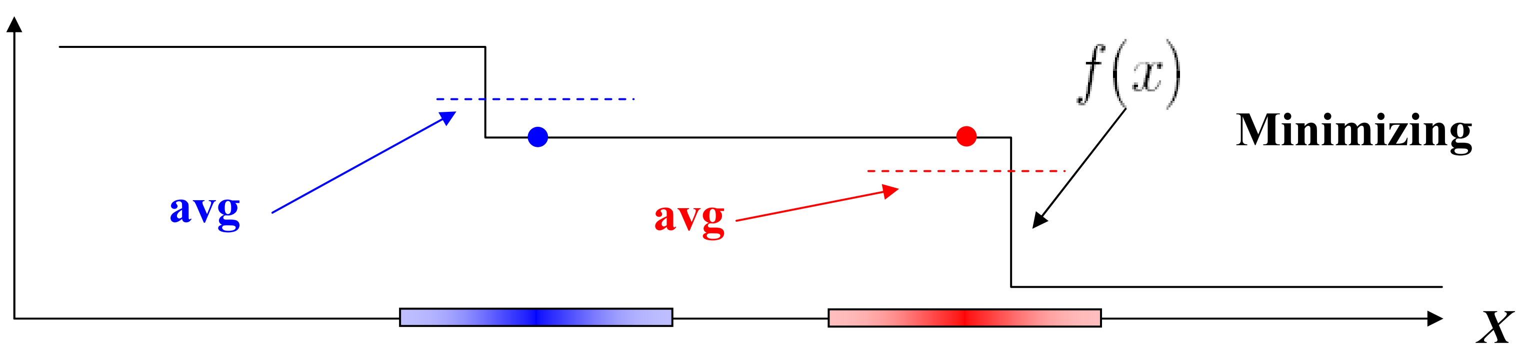
$$\operatorname*{avg}_{y \in N(x)}\{f(y)\} = \alpha f(x) + \beta \quad \forall x \in X$$

- **However, they are not the same in non-elementary landscapes. If we have *n* elementary components, then:**

$$\operatorname*{avg}_{y \in N(x)}\{f(y)\} = \alpha_0 + \alpha_1 f(x) + \sum_{i=2}^{n} \alpha_i f_i(x) \quad \forall x \in X$$

Elementary components

- **The new selection strategy could be useful for plateaus**



Autocorrelation

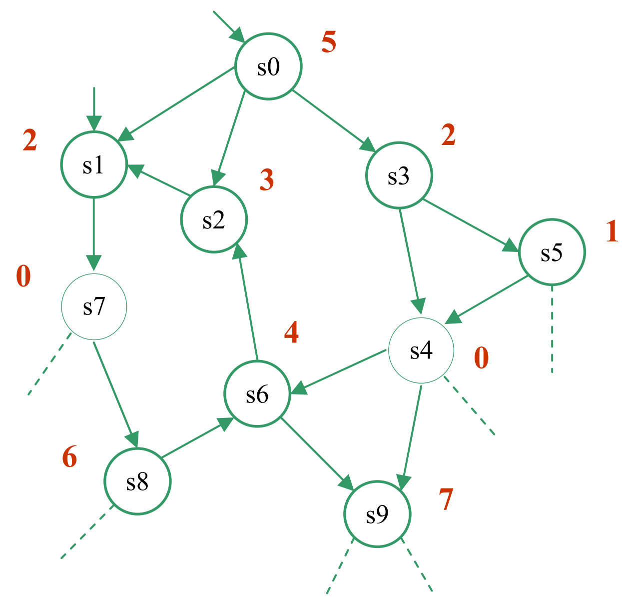
- Let $\{x_0, x_1, \ldots\}$ a simple random walk on the configuration space where $x_{i+1} \in N(x_i)$
- The random walk induces a time series $\{f(x_0), f(x_1), \ldots\}$ on a landscape.
- The autocorrelation function is defined as:

$$r(s) = \frac{\langle f(x_t) f(x_{t+s}) \rangle_t - \langle f \rangle^2}{\langle f^2 \rangle - \langle f \rangle^2}$$

- The autocorrelation length is defined as:

$$l = \sum_{s=0}^{\infty} r(s)$$

- Autocorrelation length conjecture:



The number of local optima in a search space is roughly

$$M \approx |X| / |X(x_0, l)|$$

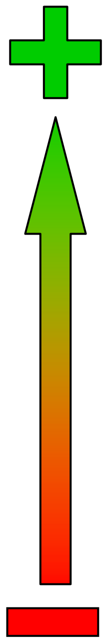
Solutions reached from x_0 after l moves

Autocorrelation Length Conjecture

- The **higher** the value of *l* the **smaller** the number of local optima and the **better** the performance of a local search method
- *l* is a measure of the **ruggedness** of a landscape

Angel, Zissimopoulos. Theoretical Computer Science 264:159-172

Length



Ruggedness	Nb. steps (config 1)		Nb. steps (config 2)	
	% rel. error	nb. steps	% rel. error	nb. steps
$10 \leq \zeta < 20$	0.2	50500	0.1	101395
$20 \leq \zeta < 30$	0.3	53300	0.2	106890
$30 \leq \zeta < 40$	0.3	58700	0.2	118760
$40 \leq \zeta < 50$	0.5	62700	0.3	126395
$50 \leq \zeta < 60$	0.7	66100	0.4	133055
$60 \leq \zeta < 70$	1.0	75300	0.6	151870
$70 \leq \zeta < 80$	1.3	76800	1.0	155230
$80 \leq \zeta < 90$	1.9	79700	1.4	159840
$90 \leq \zeta < 100$	2.0	82400	1.8	165610

Autocorrelation and Landscapes

- **If f is a sum of elementary landscapes:**

Fourier coefficients

$$r(s) = \sum_{i \neq 0} \frac{a_i^2}{\sum_{j \neq 0} a_j^2} \left(1 - \frac{k_i}{d}\right)^s \qquad l = \sum_{i \neq 0} \frac{d \ a_i^2}{k_i \sum_{j \neq 0} a_j^2}$$

- **For elementary landscapes:**

$$r(s) = \left(1 - \frac{k}{d}\right)^s \qquad l = \frac{d}{k}$$

- **Using the landscape decomposition we can determine *a priori* the performance of a local search method**

Conclusions & Future Work

Conclusions

- Elementary landscape decomposition is a useful tool to understand a problem
- The decomposition can be used to design new operators
- We can exactly determine the autocorrelation functions
- It is not easy to find a decomposition in the general case

Future Work

- Methodology for landscape decomposition
- Search for additional applications of landscapes' theory in EAs
- Design new operators and search methods based on landscapes' information

Thanks for your attention !!!