Complexity and Reducibility of the Skip Delivery Problem

C. Archetti ‡ R. Mansini † M.G. Speranza ‡

February 20, 2003

Abstract

In the Skip Delivery Problem (SDP) a fleet of vehicles must deliver skips to a set of customers. Each vehicle has a maximum capacity of two skips and has to start and end its tour at a central depot. The demand of each customer can be greater than the capacity of the vehicles. The objective is to minimize the cost of the total distance traveled by the vehicles to serve all the customers. We show that the SDP is solvable in polynomial time, while its generalization to the case where all vehicles have a capacity greater than two, known as the Split Delivery Vehicle Routing Problem, is shown to be NP-hard, even under restricted conditions on the costs. We also show that, if the costs are symmetrical and satisfy the triangle inequality, the SDP is reducible in polynomial time to a problem of possibly smaller size where each customer has unitary demand. This property allows a remarkable simplification of the problem.

Keywords: Skip Delivery Problem; Split Delivery Vehicle Routing Problem; triangle inequality; computational complexity.

Introduction

In this paper we consider the Skip Delivery Problem (SDP) where a fleet of vehicles have to deliver skips to a set of customers. Each vehicle has a maximum capacity of two skips, which is often the case in practical applications due to the size of a skip. Each vehicle must start and end its tour at a central depot and the objective is to minimize the cost of the total distance traveled by the vehicles to serve all the customers. The demand of each customer can be greater than the capacity of the vehicles. Thus, each customer can be visited more than once, contrary to the usual assumption in the Vehicle Routing Problems (VRPs) (for a survey on vehicle routing problems see (Toth and Vigo, 2002) and (Golden and Assad, 1988)).

The SDP has a number of real applications in the field of distribution or collection of goods transported by means of big containers (skips). A typical application is the selective waste collection problem, where each skip contains only one type of waste and has to be transported from a collection area to the corresponding landfill for disposal. A real application on skip waste collection is described in (Archetti and Speranza, 2002) where the authors propose a heuristic algorithm tested on real data. Another typical application is the distribution of cereals which is often carried out by means of skips. A problem of distribution of skips is also studied in (De Meulemeester, Laporte, Louveaux and Semet, 1997), where the authors consider a pick-up and delivery problem in which each vehicle can transport only one

[‡]Department of Quantitative Methods, University of Brescia, C.da S. Chiara 50,
I-25122 Brescia, Italy, archetti,speranza@eco.unibs.it

[†]Department of Electronics for Automation, University of Brescia, via Branze 38,
I-25123 Brescia, Italy, rmansini@ing.unibs.it

skip at a time. A similar problem, referred to as the Rollon-Rolloff Vehicle Routing Problem (RRVRP), is analyzed in (Ball, Bodin, Baldacci and Mingozzi, 2000). In the RRVRP a fleet of tractors have to move trailers from given locations to a disposal facility and each tractor has a capacity of exactly one trailer. At the disposal plant, full trailers are unhooked and empty trailers are hooked on the tractors. The objective is to find the routes that serve all the trailers while minimizing the total traveling time and the number of vehicles used.

The Split Delivery Vehicle Routing Problem (SDVRP) is a VRP where each customer can be visited more than once, even when the demand of the customers is smaller than the capacity of the vehicles. The SDVRP is a generalization of the SDP to the case of general values of the vehicle capacity. The SDVRP has been studied in (Dror and Trudeau, 1989 and 1990) where the authors have analyzed the savings generated by allowing split deliveries in a VRP and shown that, when the costs satisfy the triangle inequality, there exists an optimal solution for the SDVRP where no pair of tours has more than one vertex in common.

In this paper we study the SDP both in the case of a homogeneous fleet of vehicles with capacity two and in the case of a mixed fleet of vehicles with capacity one and two. The case of a homogeneous fleet of vehicles with capacity one is clearly trivial, as the only possible solution is that the vehicles make direct trips from/to the depot, by carrying one skip at a time. We show that the SDP is solvable in polynomial time while the SDVRP with vehicle capacity larger than two is NP-hard, even under restricted conditions on the costs. Finally, we introduce the property of problem reducibility. The problem is reducible when an optimal solution exists in which, for each customer, as many as possible full load direct trips from/to the depot are made till the demand of each customer is lower than the vehicle capacity. We show that, if the costs are symmetrical and satisfy the triangle inequality, the SDP is reducible in polynomial time to a problem of possibly smaller size where each customer has unitary demand. This causes a remarkable simplification of the problem.

In Section 1 we introduce the SDP and analyze the computational complexity of the SDP and the SDVRP. The results related to the property of problem reducibility are presented in Section 2.

1 Problem definition and complexity

The SDP can be defined on a graph $G = (V, E)$ where the set of vertices V represents the depot, namely vertex 1, and the customers, vertices $2, ..., n$. An integer value q_i is associated to each vertex $i \in V - \{1\}$ and represents the demand of customer i. A value c_{ij} is associated to each edge $(i, j) \in E$ and represents the cost to travel from vertex i to vertex j. A fleet of vehicles is available. We assume that the number of vehicles is unlimited. Each vehicle has to start from the depot and return to the depot. The objective is to minimize the total cost of the distance traveled by the vehicles to serve all the customers.

We consider both the case of a homogeneous fleet of vehicles with capacity two and of a mixed fleet of vehicles with capacity one and two. The case of a homogeneous fleet of vehicles with capacity one is clearly trivial, as the only possible solution is that the vehicles make direct trips from the depot, carrying one skip at a time. We refer to the problem where a homogeneous fleet of vehicles with capacity two is available as *SDP with* $Q = 2$. In the case of a mixed fleet with unlimited number of vehicles with capacity one and two, if the cost c_{ij} does not depend on the capacity of the vehicle used, clearly the problem is equivalent to the previous one with homogeneous fleet since no vehicle with capacity one will be used. Thus, the case of mixed fleet becomes interesting only if the cost c_{ij} depends on the capacity of the vehicle used. We refer to this problem as *SDP with* $Q = 1$ *or* 2 *and variable* c_{ij}. It is reasonable to assume that the cost c_{ij} is proportional to a coefficient which represents the operational cost of the vehicle per unit of distance traveled and thus depends on the type of vehicle used. Let k_1 and k_2 denote the operational coefficients for vehicles with capacity one and two, respectively. Then, the cost c_{ij} to travel from i to j for a vehicle with capacity one is $k_1\bar{c}_{ij}$, while for a vehicle with capacity two is $k_2\bar{c}_{ij}$, where $\bar{c}_{ij}$ is the distance from i to j.

We recall that the costs are symmetrical when $c_{ij} = c_{ji}$, $\forall i, j$, and satisfy the triangle inequality when $c_{ij} \leq c_{ik} + c_{kj}$, $\forall i, j, k$. We also consider, for the case of symmetrical costs, a more restrictive condition on the costs, namely the sharpened triangle inequality. The costs are said to satisfy the sharpened triangle inequality, for some $\alpha \in [\frac{1}{2}, 1]$, if:

$$c_{ij} \leq \alpha(c_{ik} + c_{kj}) \quad \forall i, j, k, i \neq j \neq k. \tag{1}$$

Note that, if the sharpened triangle inequality holds for any value of α, then also the usual triangle inequality holds. From (1)

$$c_{ij} \leq \frac{\alpha}{1-\alpha} c_{ik} \quad \forall i, j, k, i \neq j \neq k \tag{2}$$

follows and thus it can be seen that the lower the value of α, the more similar to each other the costs c_{ij}. If $\alpha = \frac{1}{2}$ the costs c_{ij} must be all identical. When $\alpha = 1$ the sharpened triangle inequality becomes the usual triangle inequality, i.e. the most frequent property of the costs observed in real applications, as costs are typically proportional to the geographical distances. The sharpened triangle inequality, with $\alpha < 1$, more frequently holds in those applications where the costs c_{ij} are proportional to the traveling times or include costs which are not proportional to the geographical distances.

1.1 Computational complexity: the SDP with general costs

In this section we study the computational complexity of the SDP with general costs in the case with $Q = 2$ and in the case with $Q = 1$ or 2 and variable c_{ij}.

In both cases we show that any instance of the SDP can be transformed into an instance of the minimum weight b-matching problem (see (Gerards, 1995)). As a polynomial algorithm is known for this problem which is a particular case of the general matching problem (see (Gerards, 1995) and (Hannemann and Schwartz, 2000)), the SDP with general costs can be solved in polynomial time.

The minimum weight b-matching problem is defined as follows (see (Gerards, 1995)). Let $\widetilde{G} = \left(\widetilde{V}, \widetilde{E}\right)$ be an undirected graph, possibly with loops, $\widetilde{c}_e$ be the weight of edge $e \in \widetilde{E}$ and $b_i \in Z^+$ a parameter associated to vertex $i \in \widetilde{V}$. The objective is to find a minimum weight integral vector $\mathbf{x} \in Z^{|\widetilde{E}|}$ such that $\sum_{e \in \delta(i)} x_e + \sum_{l \in \lambda(i)} 2x_l \geq b_i$, $i \in \widetilde{V}$, where $\delta(i)$ and $\lambda(i)$ are the set of edges incident to i and the set of loops on i, $i \in \widetilde{V}$, respectively.

Theorem 1 *The SDP with $Q = 2$ can be solved in polynomial time, in the case of general costs.*

Proof. Given an instance of the SDP with $Q = 2$ defined on graph $G = (V, E)$, we construct a corresponding instance of the minimum weight b-matching problem as follows. We construct an undirected complete graph $\widetilde{G} = \left(\widetilde{V}, \widetilde{E}\right)$ where $\widetilde{V} = V - \{1\}$, $\widetilde{c}_{ij} = min\{c_{1i} + c_{ij} + c_{j1}, c_{1j} + c_{ji} + c_{i1}\}$, $i \neq j$, $i, j \in \widetilde{V}$ and we set $b_i = q_i$, $i \in \widetilde{V}$. Then, a loop is added on each vertex

$i \in \widetilde{V}$ with cost on the loop $\widetilde{c}_{ii} = c_{1i} + c_{i1}$.

The optimal solution of the SDP with $Q = 2$ in the case of general costs on G can be obtained by solving the minimum weight b-matching problem on graph $\widetilde{G}$. The value of x on edge (i, j) gives the number of trips which visit vertices i and j. The value of x on a loop on vertex i gives the number of direct trips from the depot to vertex i. If the sum of the x of the edges incident to i and of twice the x of the loops on i exceeds q_i, it exceeds q_i by one unit and this means that one of the vehicles which visit i carries one skip only.

□

The same result holds for the SDP with $Q = 1$ or 2 and variable c_{ij}.

Theorem 2 *The SDP with $Q = 1$ or 2 and variable c_{ij} can be solved in polynomial time, in the case of general costs.*

Proof. Given an instance of the SDP with $Q = 1$ or 2 and variable c_{ij} defined on graph $G = (V, E)$, we construct an instance of the minimum weight b-matching problem as follows. We construct an undirected complete graph $\widetilde{G} = \left(\widetilde{V}, \widetilde{E}\right)$ where $\widetilde{V} = V$, $\widetilde{c}_{ij} = min\{k_2(\bar{c}_{1i} + \bar{c}_{ij} + \bar{c}_{j1}), k_2(\bar{c}_{1j} + \bar{c}_{ji} + \bar{c}_{i1})\}$, $i \neq j$, $i, j \in \widetilde{V} - \{1\}$ and $\widetilde{c}_{1i} = k_1(\bar{c}_{1i} + \bar{c}_{i1})$, $i \in \widetilde{V} - \{1\}$. We set $b_i = q_i$, $i \in \widetilde{V} - \{1\}$ and $b_1 = 0$ and add a loop on each vertex $i \in \widetilde{V} - \{1\}$ with $\widetilde{c}_{ii} = k_2(\bar{c}_{1i} + \bar{c}_{i1})$. Then, the optimal solution of the SDP with $Q = 1$ or 2 and variable c_{ij} with general costs on G can be obtained by solving the minimum-weight b-matching problem on graph $\widetilde{G}$. The value of x on edge (i, j), $i, j \neq 1$, gives the number of trips which visit vertices i and j with a vehicle with capacity two. The value of x on a loop on vertex i gives the number of direct trips from the depot to vertex i with a vehicle with capacity two. Finally, the value of x on edge $(1, i)$ gives the number of trips to i with vehicles with capacity one. □

1.2 Computational complexity: the SDVRP

Let us consider the generalization of the SDP to the case in which all the vehicles have capacity $Q \geq 3$. In the following, we show that the decision version of the SDVRP is NP-complete even if the costs are symmetrical and satisfy the sharpened triangle inequality with $1/2 < \alpha \leq 1$. The case with $\alpha = 1/2$ is clearly trivial. The decision version of the SDVRP is formulated as follows.

INSTANCE: Graph $G = (V, E)$, $V = \{1, ..., n\}$, costs $c_{ij} \in Z^+$, $(i, j) \in E$, weights $q_i \in Z^+$, $i \in V - \{1\}$ (which represent the demand of each vertex except the depot), capacity $Q \in Z^+$ and positive integer B.

QUESTION: Is there a set of tours, starting and ending at the depot, which satisfy the demand of all the vertices and the capacity constraint in all tours, such that the total cost of the tours is not greater than B?

Theorem 3 *The decision version of the SDVRP where each customer has unitary demand, with symmetrical costs that satisfy the sharpened triangle inequality with* $1/2 < \alpha \leq 1$ *is NP-complete for* $Q \geq 3$.

Proof. We consider a variant of the *Partition into Isomorphic Subgraphs Problem* (PIIS) which is stated as follows (Garey and Johnson, 1979):

INSTANCE: Graphs $\bar{G} = (\bar{V}, \bar{E})$ and $H = (V', E')$ with $\left|\bar{V}\right| = q\left|V'\right|$ for some $q \in Z^+$.

QUESTION: Can the vertices of $\bar{G}$ be partitioned into q disjoint sets $V_1, V_2, ..., V_q$ such that, for $1 \leq i \leq q$, the subgraph of $\bar{G}$ induced by V_i contains a subgraph isomorphic to H?

It has been shown (Garey and Johnson, 1979) that this variant of the PIIS remains NP-complete for any fixed H which contains a connected component of three or more vertices. In the following we show a polynomial transformation from this variant of the PIIS, with H a path of Q vertices, $Q \geq 3$ (from now on simply PIIS-path problem), to the SDVRP with capacity Q. Let us consider an instance of the PIIS-path problem. We now construct the corresponding instance of the SDVRP. Let G be a complete graph with $V = \bar{V} \cup \{1\}$, where vertex 1 is the depot. Thus $n = qQ + 1$. We take $q_i = 1$, $i \in \bar{V}$. The costs are symmetrical and we set, for $i, j \in \bar{V}$ and $0 < \varepsilon \leq 2\alpha - 1$,

$$c_{ij} = \begin{cases} 1 \text{ if } (i,j) \in \bar{E} \\ 1 + \varepsilon \text{ if } (i,j) \notin \bar{E}. \end{cases}$$

Moreover, we set $c_{1i} = C = 1 + \varepsilon$, $i \in \bar{V}$. Clearly, c_{ij}, $i, j \in V$, satisfy the sharpened triangle inequality with $1/2 < \alpha \leq 1$. Finally, we take $B = (Q - 1 + 2C)q$. The minimum number of tours to serve the vertices in $\bar{V}$ is q, thus the cost due the edges that are incident to vertex 1 in the solution of the SDVRP is at least $2Cq$. The minimum number of edges to be taken to serve the vertices in $\bar{V}$ is $q(Q + 1)$. Therefore, the minimum cost to serve the vertices in $\bar{V}$ is $(Q - 1 + 2C)q = B$. The solutions which make more than q tours, e.g. $q + d$ tours, $d \in Z^+$, have a cost of $2C(q + d)$ for the edges that are incident to the depot and a cost of at least $q(Q - 1) - 2d$ for the remaining edges (since the total minimum number of edges is $q(Q + 1)$). Thus the total cost of these solutions is greater than or equal to $2C(q + d) + q(Q - 1) - 2d = q(Q - 1 + 2C) + 2d(C - 1) > B$. This implies that we need only to consider solutions with q tours, where each tour visits Q vertices (if a tour visits less than Q vertices then there exists a tour which visits more than Q vertices and this violates the capacity constraint). An instance of the SDVRP has 'yes' answer if and only if there exist q paths, which visit Q vertices each, cover all the vertices in $\bar{V}$ and have total cost $B' = B - 2Cq = (Q-1)q$. Such a solution exists if and only if each of the q paths has cost lower than or equal to $Q-1$. Since all the edges of G have cost greater than or equal to 1 each path must have a cost of $Q-1$, that is all edges in each path must have cost equal to 1. We can now see that the instance of the PIIS-path problem has 'yes' answer if and only if the corresponding instance of the SDVRP has 'yes' answer. If the PIIS-path problem has 'yes' answer, then there exists a partition of $\bar{V}$ into q sets such that the subgraph of $\bar{G}$ induced by each set contains a path of Q vertices. Since these paths identify q paths of cost $Q-1$ each in G, the SDVRP has 'yes' answer. Viceversa, if the SDVRP has 'yes' answer, then there exist q paths of $Q-1$ edges of cost 1 each. This implies that these edges are edges of $\bar{E}$ and then the PIIS-path problem has 'yes' answer. As this is a polynomial transformation, the decision version of the SDVRP is NP-complete. □

The NP-hardness of a particular case of the VRP, that is the problem where each customer with given demand must be visited exactly once by an unlimited fleet of capacitated vehicles with the objective of minimizing the total cost of the routes, follows directly from Theorem 3.

Remark 1 *The VRP with unitary demands and symmetrical costs that satisfy the sharpened triangle inequality with* $1/2 < \alpha \leq 1$ *is NP-hard when the capacity of the vehicles is greater than or equal to* 3.

Obviously, all the more general cases, such as those with costs that satisfy the triangle inequality, are NP-hard.

2 Reducibility of the Skip Delivery Problem

An instance of the SDP is *reducible* if there exists an optimal solution in which each vertex is served by as many full load depot/vertex direct trips as possible by vehicles with capacity two. Then, the vertices with an even number of skips will be completely served by the direct trips, while one skip will remain in the other vertices.

Definition 1 *A SDP instance is reducible if an optimal solution exists such that each vertex that contains two or more skips is served by as many direct trips as possible from the depot to the vertex, which deliver two skips in each trip, until the demand of each vertex is lower than two.*

When an instance of the problem is reducible, we call *reduced* the instance which is obtained by changing the demand q_i of customer i with $(q_i \ mod \ 2)$ and deleting the vertices, and related arcs, when $(q_i \ mod \ 2) = 0$. The reduction of the original instance requires a linear time in the number of the vertices.

In this section we prove that, if the costs are symmetrical and satisfy the triangle inequality, the SDP with $Q = 2$ and the SDP with $Q = 1$ or 2 and variable c_{ij} with $1 < \frac{k_2}{k_1} < 2$ are reducible. We also show that these conditions are not sufficient to make the SDVRP reducible.

2.1 The SDP with $Q = 2$

We analyze the SDP with $Q = 2$ in the cases where the costs satisfy or do not satisfy the triangle inequality. We show that the particular case in which the costs are symmetrical and satisfy the triangle inequality is reducible.

Theorem 4 *If the costs are symmetrical and satisfy the triangle inequality, the SDP with* $Q = 2$ *is reducible.*

Proof. If all vertices have unitary demand there is nothing to prove. Suppose that at least one vertex, say i, exists with demand greater than one and consider a solution in which two skips of i are delivered by two vehicles. If there is a vehicle which delivers only one skip to i and then returns to the depot, then, because of the triangle inequality, a not worse solution is to deliver the other skip of i with this vehicle. Thus we consider the case where there are two vehicles which visit i and a different vertex. Let us suppose that the first vehicle delivers one skip to i and one skip to a vertex j. The second vehicle delivers one skip to i and one skip to a vertex z (which may coincide with j). The cost of this solution is

$$z_1 = C + c_{1i} + c_{ij} + c_{j1} + c_{1i} + c_{iz} + c_{z1}, \tag{3}$$

where C represents the cost of delivering all the other skips.

Then there exists a solution in which the two skips of i are delivered together with the first vehicle, the skips in j and z are delivered with the second vehicle and all the other tours are not changed. The cost of this solution is

$$z_2 = C + c_{1i} + c_{i1} + c_{1j} + c_{jz} + c_{z1}. \tag{4}$$

As the costs are symmetrical and $c_{ij} + c_{iz} \geq c_{jz}$ for the triangle inequality, then $z_2 \leq z_1$.

As the above argument can be applied to all pairs of skips of the same vertex and to all vertices with demand greater than one, the solution in which full load direct trips are made from the depot to each vertex until the demand of each vertex is zero or one is not worse than any other solution. Thus the SDP with $Q = 2$ and symmetrical costs which satisfy the triangle inequality is reducible. □

The reduced instance of the SDP where $Q = 2$, which is a VRP with a homogeneous fleet of vehicles of capacity two, can be transformed into an instance of the generalized minimum cost matching problem. This result follows from (Christofides, 1985), where it is shown that an instance of the VRP with a homogeneous fleet of vehicles where the sum of the three smallest customer demands exceeds the capacity of the vehicles is solvable as a generalized minimum cost matching problem. A similar case has been studied in (Gourdin, Labbè and Laporte, 2000) where the authors consider the Uncapacitated Facility Location Problem with Client Matching which is a location problem where the customers assigned to each facility can be served separately or matched to another customer in a single tour.

The reducibility property does not hold for the more general case of asymmetrical costs which satisfy the triangle inequality, as shown by the following example.

Example 1 Let the graph of Figure 1 represent an instance of the SDP, where $q_2 = 2$, $q_3 = q_4 = 1$ and the weights on the arcs represent the costs which are asymmetrical and satisfy the triangle inequality. Then the solution in which the first vehicle delivers one skip to vertex 2 and one skip to vertex 3 and the second vehicle delivers one skip to vertex 2 and one skip to vertex 4 has a cost of 8 (as the costs are asymmetrical, the order of visit of the vertices is important and we consider the one that gives the minimum cost). The solution in which only vertex 2 is visited in one tour and then vertices 3 and 4 are visited in the second tour has a cost of 9. Thus the first solution is cheaper (it is also the optimal solution) and the problem is not reducible.

Insert here Figure 1

Note that the definition of reducibility can be extended to the SDVRP. The following example shows that, for the SDVRP with capacity $Q \geq 3$, the condition of symmetrical costs which satisfy the triangle inequality is not sufficient to make the problem reducible.

Example 2 Let $G = (V, E)$, $|V| = 5$, be a graph representing a SDVRP where vertex 1 is the depot, $q_2 = Q$, $q_3 = q_4 = Q - 1$, $q_5 = 2$ and the costs are symmetrical being all equal to two but for the edges incident to vertex 2 where the costs are equal to one. Then, the cost of the solution in which three tours are made which visit vertex 2 and, respectively, vertices 3, 4 and 5 is 12. The cost of all possible solutions that deliver the Q skips to vertex 2 in one tour and then deliver the remaining skips of vertices 3, 4 and 5 in two (or three) tours is 14. Thus, the first solution is cheaper (and it is also the optimal solution) and the SDVRP with $Q \geq 3$ is not reducible. Note that if $Q = 2$ the instance is reducible.

Also in the particular case in which the costs are Euclidean, there exist instances for which the SDVRP with $Q \geq 3$ is not reducible, as shown by Example 3.

Example 3 Let $\varepsilon > 0$, $k > 0$. Let vertices $2, 3$ and 4 be placed on a circle of radius 2ε while vertex 5 is placed in the center of the circle and let $q_2 = Q$, $q_3 = q_4 = Q - 1$ and $q_5 = 2$ (see Figure 2). Vertices 3 and 4 have both cost equal to ε from vertex 2 and the cost from vertex 2 to the depot is equal to k. Let θ be the angle insisting on the chord $(3, 2)$. With some simple calculus, we can obtain the costs between vertices 3 and 4 and between vertex 3 (or 4) and the depot. Noting that the triangle formed by the vertices $3, 2$ and 5 is isosceles with basis $(3, 2)$ we have $\theta \simeq 28.96°$. Thus we obtain $c_{34} = 4\varepsilon \sin\theta \simeq 1.94\varepsilon$, $c_{h2} = \sqrt{\varepsilon^2 - (2\varepsilon \sin\theta)^2} = \frac{1}{4}\varepsilon$. Then, $c_{h1} = k + \frac{1}{4}\varepsilon$ and $c_{31} = c_{41} = \sqrt{(2\varepsilon\sin\theta)^2 + \left(k + \frac{1}{4}\varepsilon\right)^2} = \sqrt{k^2 + \varepsilon^2 + \frac{1}{2}k\varepsilon}$. The cost of the solution that makes three tours which visit vertex 2 and, respectively, vertices $3, 4$ and 5 is $z_1 = 4k + 2c_{31} + 6\varepsilon$. The lowest cost of the solution that delivers Q skips of vertex 2 in one tour is the solution that visits vertex 3 (or 4) and vertex 5 in the second tour and vertex 3 (or 4) and vertex 4 (or 3) in the third tour. The cost of this solution is $z_2 = 3k + 3c_{31} + 4\varepsilon + c_{34}$.

By subtracting z_2 from z_1 we obtain

$$z_1 - z_2 = k + 0.06\varepsilon - \sqrt{k^2 + \varepsilon^2 + \frac{1}{2}k\varepsilon},$$

which is negative for $\varepsilon > -\frac{0.38}{0.9964}k$ and thus for any ε and k. Then the first solution is cheaper than the second one (and it is also the optimal solution) and the SDVRP with $Q \geq 3$ is not reducible.

Insert here Figure 2

In (Archetti, Mansini and Speranza, 2001) it is shown that the SDVRP with $Q = 3$ is reducible when the costs are symmetrical and satisfy the sharpened triangle inequality with $\alpha \leq \frac{2}{3}$.

2.2 The SDP with $Q = 1$ or 2 and variable c_{ij}

In this section we show that, if the costs are symmetrical and satisfy the triangle inequality, the SDP with $Q = 1$ or 2 and variable c_{ij} is reducible. If $\frac{k_2}{k_1} \leq 1$, then the unitary operational cost of vehicles with capacity two is lower than or equal to the cost of vehicles with capacity one and thus there exists an optimal solution which only uses vehicles with capacity two. If $\frac{k_2}{k_1} \geq 2$, the symmetry of the costs implies that there exists an optimal solution which only uses vehicles with capacity one. The interesting case is $1 < \frac{k_2}{k_1} < 2$.

Theorem 5 *If the costs are symmetrical and satisfy the triangle inequality, the SDP with $Q = 1$ or 2 and variable c_{ij}, with $1 < \frac{k_2}{k_1} < 2$, is reducible.*

Proof. Let us suppose that at least one vertex, say i, has a demand of two skips. As $1 < \frac{k_2}{k_1} < 2$, the solution which serves vertex i with a vehicle with $Q = 2$ is better than the solution which uses two vehicles with $Q = 1$. Moreover, using the same argument used in the proof of Theorem 4, we can show that there always exists a solution in which the two skips of i are delivered with a direct trip from the depot which is not worse than any other solution.

As the above argument can be used for all pairs of skips of the same vertex, the solution in which full load direct trips are made from the depot to all vertices with vehicles with capacity two until the demand of the vertices is zero or one is not worse than any other solution. Thus, if the costs are symmetrical and satisfy the triangle inequality, the SDP with $Q = 1$ or 2 and variable c_{ij} is reducible. $\square$

The reduced instance of the SDP with $Q = 1$ or 2 and variable c_{ij} can be transformed into an instance of the generalized minimum cost matching problem. The procedure used to show this result is similar to the procedure used in (Christofides, 1985) for the VRP with a homogeneous fleet of vehicles where the sum of the three smallest customer demands exceeds the capacity of the vehicles, cited in Section 2.1.

Let $G' = (V', E')$ be a complete graph where $V' = V - \{1\}$ and the cost c'_{ij} of each edge $(i, j) \in E'$ is

$$c'_{ij} = k_2(\bar{c}_{1i} + \bar{c}_{ij} + \bar{c}_{j1}).$$

We assign a penalty $p_i = k_1 2\bar{c}_{1i}$ to each vertex i, $i \in V'$. The optimal solution of the reduced instance can be obtained by solving a generalized minimum cost matching problem on G' where the objective is to find a matching such that the sum of cost of the edges in the matching plus the sum of the penalties of the vertices that are unmatched is minimum. In the graph G', the matching of vertex i to vertex j is interpreted as a route $(1, i, j, 1)$ with a vehicle of capacity two in the SDP with $Q = 1$ or 2 and variable c_{ij}. A vertex i left unmatched is interpreted as a route $(1, i, 1)$ with a vehicle of capacity one.

3 Conclusions

In this paper we have analyzed the Skip Delivery Problem where a fleet of vehicles of maximum capacity equal to two must deliver skips to a set of customers and each customer has a demand possibly larger than the vehicle capacity. We have shown that the problem is solvable in polynomial time while its extension to the case with capacity greater than two, known as the Split Delivery Vehicle Routing Problem, is NP-hard even under restricted conditions on the costs. We have proved that, if the costs are symmetrical and satisfy the triangle inequality, the Skip Delivery Problem is reducible in polynomial time to a new problem where each customer has unitary demand, with a possible reduction of the number of customers. This reduction causes a remarkable simplification of the problem.

An open question concerns the existence of conditions that allow to reduce the problem in the case of asymmetrical costs.

Acknowledgments

We wish to thank E. Angelelli and C. Arbib for useful discussions at an early stage of this work and A. Schwartz for fruitful information on the b-matching problem. Thanks are also due to the referees and the editor for their valuable comments.

References

[1] C. Archetti, R. Mansini, M.G. Speranza, "The split delivery vehicle routing problem with small capacity", Technical Report n. 201, Department of Quantitative Methods, University of Brescia (2001).

[2] C. Archetti, M.G. Speranza, "The 1-skip collection problem", Technical Report n. 209, Department of Quantitative Methods, University of Brescia (2002).

[3] M. Ball, L. Bodin, R. Baldacci, A. Mingozzi, "The rollon-rolloff vehicle routing problem", *Transportation Science* 34, 271–288 (2000).

[4] N. Christofides, "Vehicle routing", in *The Traveling Salesman Problem*, E.L. Lawler, J.K. Lenstra, A.H.G. Rinnoy Kan, D.B. Shmoys, (eds) Wiley, Chichester (1985).

[5] N. Christofides, A. Mingozzi, P. Toth, "The vehicle routing problem", in *Combinatorial Optimization*, N. Christofides, A. Mingozzi, P. Toth, C. Sandi, (eds) Wiley, Chichester (1979).

[6] L. De Meulemeester, G. Laporte, F.V. Louveaux, F. Semet, "Optimal scheduling of skip collections and deliveries", *Journal of the Operational Research Society* 48, 57–64 (1997).

[7] M. Dror, P. Trudeau, "Savings by split delivery routing", *Transportation Science* 23, 141–145 (1989).

[8] M. Dror, P. Trudeau, "Split delivery routing", *Naval Research Logistics* 37, 383–402 (1990).

[9] M.R. Garey, D.S. Johnson, "*Computers and Intractability: a Guide to the Theory of NP-Completeness*", Freeman, New York, NY (1979).

[10] A.M.H. Gerards, "Matching", in *Handbooks in Operations Research and Management Science*, vol. 7, M.O. Ball, T.L. Magnanti, C.L. Monma, G.L. Nemhauser, (eds) North-Holland, Amsterdam (1995).

[11] B.L. Golden, A.A. Assad, "*Vehicle Routing: Methods and Studies*", North-Holland, Amsterdam (1988).

[12] É. Gourdin, M. Labbé, G. Laporte, "The uncapacitated facility location problem with client matching", *Operations Research* 48, 671–685 (2000).

[13] M.M. Hannemann, A. Schwartz, "Implementing weighted b-matching algorithms: towards a flexible software design", *ACM Journal of Experimental Algorithmics* 5, article 8 (2000).

[14] P. Toth, D. Vigo, "*The Vehicle Routing Problem*", SIAM Monographs on Discrete Mathematics and Applications, Philadelphia (2002).