Solving Stochastic Routing Problems with the Integer L-Shaped Method

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Abstract

In recent years, several advances have been made towards the solution of stochastic vehicle routing problems (SVRPs). In particular, the Integer L-Shaped Method has been used for the exact solution of several families of SVRPs. In this article, the SVRP with recourse is first formulated. The Integer L-Shaped Method is outlined and implementation results on four different SVRPs are reported.

Key words: Stochastic vehicle routing problem, Integer L-shaped method.

Résumé

Au cours des dernières années, on a assisté à plusieurs développements reliés à la résolution de problèmes de tournées de véhicules stochastiques (PTVS). En particulier, l'emploi de la méthode "Integer L-Shaped" a permis la résolution exacte de plusieurs familles de PTVS. Dans cet article, on formule le PTVS avec recours et on décrit brièvement la méthode "Integer L-Shaped". On montre ensuite comment cette méthode a été appliquée à quatre PTVS différents.

Mots-clefs: Problèmes de tournées de véhicules stochastiques, méthode "Integer L-shaped".

1 Introduction

The deterministic *Vehicle Routing Problem* (VRP) is defined on a graph $G = (V, E)$, where $V = \{v_1, \ldots, v_n\}$ is a vertex set, and $E = \{(v_i, v_j) : v_i, v_j \in V, i < j\}$ is an edge set. Vertex v_1 represents a depot at which are based m identical vehicles of capacity D, while the remaining vertices are customers. A travel cost matrix $C = (c_{ij})$ and a travel time matrix $T = (t_{ij})$ are defined on E. Since the problem is defined on an edge structure, C and T are symmetrical. There exist, however, asymmetric versions of the VRP defined on an arc structure arising, for example in urban contexts with several one-way streets. The VRP consists of designing a set of m least cost vehicle routes starting and ending at the depot, such that each customer is visited exactly once, and satisfying some side constraints. We consider the following constraints.

i) **Capacity constraints.** With each customer v_i is associated a demand d_i. All demands are either collected or delivered, but not both. Then, the total demand of each vehicle route may not exceed D.

ii) **Time constraints.** There is a service time s_i associated with each customer. Then the total duration of each route, including service and travel times, may not exceed a given bound B.

For a recent survey and a bibliography on the deterministic VRP, see Fisher (1995), and Laporte (1997).

In several practical contexts, one or several components of the problem are random. This gives rise to a *Stochastic Vehicle Routing Problem* (SVRP). Here are three common examples:

i) **Stochastic customers.** Each customer v_i is present with probability p_i and absent with probability $1 - p_i$.

ii) **Stochastic demands.** The demand d_i of each customer is a random variable.

iii) **Stochastic times.** Service times s_i and travel times t_{ij} are random variables.

When some data are random, it is no longer possible to require that all constraints be satisfied for all realizations of the random variables. It follows that the decision maker may either require the satisfaction of some constraints with a given probability, or the incorporation into the model of corrective actions to be taken when a constraint is violated. The first case is referred to as *Chance Constrained Programming* (CCP). In the VRP with stochastic demands or times, the chance constrained model is relatively simple to solve as it can often be modeled as a deterministic VRP (Stewart and Golden, 1983; Laporte, Louveaux and Mercure 1989, 1992). In the second case, the SVRP can be cast as a *Stochastic Program with Recourse* (SPR). Here, a *first stage solution* is determined before knowing the realizations of the random variables. In a second stage, a recourse or corrective action can then be taken. For example, in the capacity constrained SVRP where demands are collected, the vehicle may return to the depot to unload as soon as it is full; in the time constrained SVRP, the duration of a route may exceed B, in which case a penalty may have to be paid. Such situations do not arise in deterministic contexts. In an SPR, the aim is to determine a solution of least expected total cost. For classical references on SVRPs, see Bertsimas (1992), Bertsimas, Jaillet and Odoni (1990), Golden and Stewart (1983), Jaillet (1988). For a recent survey, see Gendreau, Laporte and Séguin (1996). For a general reference on stochastic programming, see the recent book by Birge and Louveaux (1997).

In practice, SVRPs are much more difficult to solve than their deterministic counterparts, but in recent years, interesting advances have been made towards their solution by exact and approximate methods. In particular, the development of the *Integer L-Shaped Method* (Laporte and Louveaux 1993) has enabled the solution of several classes of SVRPs with recourse. The objective of this paper is to report on these developments. The remainder of this paper is organized as follows. In Section 2, we provide a general formulation of SVRPs with recourse. The Integer L-Shaped Method is summarized in Section 3. In Section 4, we report on implementations and results obtained on four different SVRPs. The conclusion follows in Section 5.

2 Formulation

The following generic two-index formulation can be applied to all SVRPs introduced in Section 1. Let x_{ij} be an integer variable equal to the number of times edge (v_i, v_j) appears in the first stage solution. If $i, j > 1$, then x_{ij} can only take the values 0 or 1, if $i = 1$, x_{ij} can also be equal to 2 if a vehicle makes a return trip between the depot and v_j. The problem is then:

$$\text{(SVRP)} \qquad \text{Minimize} \sum_{i<j} c_{ij} x_{ij} + Q(x) \tag{1}$$

subject to

$$\sum_{j=2}^{n} x_{1j} = 2m \tag{2}$$

$$\sum_{i<k} x_{ik} + \sum_{j>k} x_{kj} = 2 \qquad (k = 2, \ldots, n) \tag{3}$$

$$\sum_{v_i, v_j \in S} x_{ij} \leq |S| - 1 \qquad (S \subset V \backslash \{v_1\}; 2 \leq |S| \leq n-2) \tag{4}$$

$$0 \leq x_{ij} \leq 1 \qquad (2 \leq i < j \leq n) \tag{5}$$

$$0 \leq x_{1j} \leq 2 \qquad (j = 2, \ldots, n) \tag{6}$$

$$x = (x_{ij}) \text{ integer }, \tag{7}$$

where $Q(x)$ is the expected second stage recourse function.

Apart from the objective function, this model is that of the *m-Traveling Salesman Problem* (m-TSP) which generalizes the classical 1-TSP formulation (Dantzig, Fulkerson and Johnson, 1954). Constraints (2) and (3) are called degree constraints; subtour elimination constraints (4) prevent the formation of subtours disconnected from the depot. Note that demands and vehicle capacity are not taken into account in this model, except implicitly in $Q(x)$. Contrary to what is done in the deterministic case (Laporte, Nobert and Desrochers, 1985), there are no constraints to prevent the formation of routes including the depot, but violating capacity or duration constraints. However, it makes sense to impose, for example, constraints specifying that the expected demand of a route does not exceed D or that its expected duration does not exceed B. Such constraints have exactly the same structure as those used in the deterministic case.

The expected recourse function $Q(x)$ is problem dependent and is also related to the particular choice of possible recourse actions. For example, in the capacity constrained SVRP with collections, a possible recourse action is to return to the depot when the vehicle is full in order to unload, and then resume collections as planned. Alternatively, the remaining part of the planned route could be reoptimized. Another policy could be to plan a preventive return to the depot even if the vehicle is not full. In such a case, this decision could depend on the amount already collected and on the distance separating the vehicle from the depot. Such recourse policies are of course quite sophisticated and rather involved. The choice of a recourse action also depends on the time at which information becomes available. Always taking the same example, a vehicle not yet full may return to the depot if it is known that going to the next customer would cause its capacity to be exceeded.

The choice of an appropriate recourse policy depends of course on the situation at hand, but also on what is computationally feasible. In general, $Q(x)$ cannot be written as a linear function of x. Moreover, in most cases it will also be impractical to derive an expression for $Q(x)$ as a mathematical integer program in terms of the x_{ij} variables. Consider for example the SVRP with stochastic times, where the recourse consists of penalties proportional to overtime on routes whose duration exceed B. If a two-index formulation is used, then there is no way of knowing to which vehicle route belongs an edge (v_i, v_j) with $x_{ij} = 1$. Hence, the penalty cannot be expressed analytically in terms of the $x_{ij}s$. This difficulty does not arise in a three-index model (with the third index specifying the vehicle associated with an edge), but only relatively small problems can be solved using such formulations (Laporte, Louveaux and Mercure, 1992). If penalties for overtime are not proportional to excess duration, then even in the case of a three-index formulation, $Q(x)$ cannot be simply formulated in terms of the x variables. In any case, $Q(x)$ typically does not have nice analytical properties such as continuity and convexity as it is the solution of a second-stage stochastic integer program (Birge and Louveaux, 1997). Note, however, that in many SVRPs, it is relatively easy to compute $Q(x)$, for a number of recourse policies, once a solution in terms of x is known. The Integer L-Shaped Method presented in the next section exploits this property.

3 The Integer L-Shaped Method

The Integer L-Shaped Method belongs to the same class as Benders' decomposition (1962) and the L-Shaped Method for continuous stochastic programming (Van Slyke and Wets, 1969). In the context of the SVRP, it operates on a so-called "current problem" (CP) obtained by relaxing (SVRP) in three ways: *i)* integrality constraints (7) are relaxed; *ii)* subtour elimination constraints (4) are relaxed; *iii)* the recourse function $Q(x)$ is replaced by a lower bound θ on its value. As in standard branch-and-cut methods, integrality is gradually reached through a branching process, and violated constraints (4) are introduced as they are found to be violated. In addition, a lower bounding constraint on θ, called "optimality cut", is introduced into (CP) at integer solutions. In other words, contrary to what is done in deterministic branch-and-cut, a cut may be introduced at a feasible integer solution as θ may be strictly less than $Q(x)$ and therefore, the objective value is not yet known and better solutions may exist down the current branch.

We now outline the Integer L-Shaped Method for the SVRP. This method assumes that given a feasible solution x, then $Q(x)$ can be computed. Moreover, a finite lower bound L on $Q(x)$ is assumed to be available.

Step 0. Set the iteration count $\nu := 0$ and introduce the bounding constraint $\theta \geq L$ to (CP). Set the value $\overline{z}$ of the best known solution equal to ∞. The only pendent node corresponds to the initial current problem.

Step 1. Select a pendent node from the list. If none exists stop.

Step 2. Set $\nu := \nu + 1$ and solve (CP). Let (x^ν, θ^ν) be an optimal solution.

Step 3. Check for any subtour elimination constraint violation using, for example, one of the heuristics described in Padberg and Grötschel (1985). If at least one violation of constraints (4) can be identified, introduce a suitable number of these constraints into (CP), and return to Step 2. Otherwise, if $cx^\nu + \theta^\nu \geq \overline{z}$, fathom the current node and return to Step 1.

Step 4. If the solution is not integer, branch on a fractional variable. Append the corresponding subproblems to the list of pendent nodes and return to Step 1.

Step 5. Compute $Q(x^\nu)$ and set $z^\nu := cx^\nu + Q(x^\nu)$. If $z^\nu < \overline{z}$, set $\overline{z} := z^\nu$.

Step 6. If $\theta^\nu \geq Q(x^\nu)$, then fathom the current node and return to Step 1. Otherwise, impose the cut

$$\theta \geq L + \left(Q(x^\nu) - L\right)\left(\sum_{\substack{1<i<j \\ x_{ij}^\nu=1}} x_{ij} - \sum_{1<i<j} x_{ij}^\nu + 1\right) \tag{8}$$

into (CP) and return to Step 2.

Most steps of this algorithm have been explained, except for constraints (8). These cuts use the fact that a feasible SVRP solution is fully characterized by the variables x_{ij} for which $i > 1$. They state that either the current solution must be maintained, in which case (8) becomes $\theta \geq Q(x^\nu)$, or a new solution must be identified, in which case (8) becomes $\theta \geq L$ or less, and thus redundant. Alternatively, one could impose

$$\sum_{\substack{1<i<j \\ x_{ij}^\nu=1}} x_{ij} \leq \sum_{1<i<j} x_{ij}^\nu - 1. \tag{9}$$

The advantage of (9) is that all its variables have a coefficient of 1 and are less likely to cause numerical problems (see, Séguin, 1994). On the other hand, constraint (8) can more easily be lifted as shown by Laporte and Louveaux (1993) and by Hjorring and Holt (1996).

Finally note that similar to what is commonly done in branch-and-cut, valid inequalities can be introduced to reinforce the current relaxed problems. In particular valid inequalities involving θ, also known as "lower bounding functionals", can sometimes be generated. An example is provided in the following section.

4 Implementation and results

We now illustrate the computation of $Q(x)$, of L, and of a lower bounding functional on two versions of the SVRP with stochastic demands. For the first two computations, we work on the standard m-vehicle SVRP. In the third example, only one vehicle is used.

4.1 Computation of $Q(x)$

In this example, there are m vehicles of capacity D, and each customer v_i has a stochastic demand d_i. A standard policy is for vehicles to follow their planned route and collect demand as long as their capacity is not exceeded. If at some point the vehicle capacity is exceeded, "failure" is then said to occur. The vehicle then returns to the depot to unload and resumes collections starting at the point of failure. The simplest case occurs when the following two conditions hold: *i)* the cumulative load of the vehicle at any point is exactly equal to D with probability 0; *ii)* the probability of having two failures on the same vehicle route is negligeable. Condition *i)* holds, for example, whenever d_i is a continuous random variable; condition *ii)* is satisfied provided that routes are planned in such a way that their total expected demand is not excessive with respect to the total vehicle capacity. Under these assumptions, Dror and Trudeau (1986) show how to compute the expected cost of recourse $Q^{k,\lambda}(x)$ for a given orientation $\lambda = 1$ or 2 of a route k, corresponding to $(v_{i_1} = v_1, v_{i_2}, \ldots, v_{i_{t+1}} = v_1)$. For $\lambda = 1$, this expected cost is

$$Q^{k,1}(x) = 2\sum_{j=3}^{t} \Big[F^{j-1}(D) - F^{j}(D)\Big] c_{1i_j}, \tag{10}$$

with $F^j(D) = P\Big(\sum_{\ell=2}^{j} d_{i_\ell} \leq D\Big)$. A similar expression can be computed for $\lambda = 2$, *mutatis mutandis.* The value of $Q(x)$ is then

$$Q(x) = \sum_{k=1}^{m} \min\{Q^{k,1}(x), Q^{k,2}(x)\}.$$

4.2 Computation of L

Consider the same problem as in Section 4.1 under the same two conditions. A valid lower bound L on $Q(x)$ is simply computed as follows. Relabel all customers so that $c_{1,i+1} \geq c_{1,i} (i = 2, \ldots, n-1)$ and let q_j be a lower bound on the probability of having at least j failures in the solution, with $j \leq m$, i.e.,

$$q_j = P\Big(\sum_{i=2}^{n} d_i > (m+j-1)D\Big) \; (j = 1, \ldots, m).$$

Then a valid lower bound on $Q(x)$ is given by

$$L = 2\sum_{j=1}^{m} c_{1,j+1}\, q_j. \tag{11}$$

The validity of (11) is established as follows. Let $f_i(x)$ be the probability of having i failures in solution x. Then

$$\begin{aligned} Q(x) &\geq 2\sum_{i=1}^{m} f_i(x) \sum_{j=1}^{i} c_{1,j+1} \\ &= 2\sum_{j=1}^{m} c_{1,j+1} \sum_{i=j}^{m} f_i(x) \\ &\geq 2\sum_{j=1}^{m} c_{1,j+1}\, q_j = L. \end{aligned}$$

4.3 Computation of a lower bounding functional on $Q(x)$

Finally consider the same problem with $m = 1$. We now present in a simplified form a lower bounding functional developed by Hjorring and Holt (1996).

Consider a partial route $h = (v_{i_1} = v_1, v_{i_2}, \ldots, v_{i_r})$, with $r \leq n-1$, and let $H = \{v_{i_1}, \ldots, v_{i_r}\}$. Create an artificial customer $v_{i_{r+1}}$ with $d_{i_{r+1}} = \sum_{v_i \in V \setminus H} d_i$ and $c_{1,i_{r+1}} = \min_{v_i \in V \setminus H} \{c_{1i}\}$. Then construct route $k = (v_{i_1}, v_{i_2}, \ldots, v_{i_r}, v_{i_{r+1}}, v_1)$, compute $Q^{k,\lambda}(x)$ for $\lambda = 1$ and 2, as in (10), and observe that $\theta_h = \min\{Q^{k,1}(x), Q^{k,2}(x)\}$ is a lower bound on $Q(x)$ for any solution including h as a partial route. Therefore

$$\theta \geq (\theta_h - L)\left(\sum_{\ell=1}^{r-1} x_{i_\ell i_{\ell+1}} - (r-2)\right) + L$$

is a valid lower bounding functional on $Q(x)$.

4.4 Available results

To our knowledge, the Integer L-Shaped Method has been applied to four different SVRPs. We summarize the available results in Table 1. These results indicate that stochastic problems can only be solved to optimality for relatively small sizes, as compared to their deterministic counterpart. The last column suggests that problem difficulty increases when the first-stage part of the objective function provides little information on the exact expected solution cost.

Table 1: Summary of computational results

Problem	*Reference*	*Computation of $Q(x)$*	*Computation of L*	*Extra cuts*	*Largest size solved*	*Main source of difficulty*
Stochastic travel times, m vehicles	Laporte, Louveaux and Mercure (1992)	Direct probabilistic computation	0	Lower bounding functional on aggregate travel time	$n = 20$	Value of penalty for excessive duration
Stochastic customers, 1 vehicle	Laporte, Louveaux and Mercure (1994)	Dynamic programming	Linear relaxation of an integer linear program	Lower bounding functional on the expected route length	$n = 50$	Number of stochastic customers
Stochastic customers and demands, m vehicles	Séguin (1994), Gendreau, Laporte and Séguin (1995)	Direct probabilistic computation on a transformed objective	0 for the transformed objective	None	$n = 46$ $n = 70$ (stochastic demands only)	Number of stochastic customers
Stochastic demands, 1 vehicle	Hjorring and Holt (1996)	Direct probabilistic computation	Partial routes	Lower bounding functional on partial routes	$n = 90$	Demand/capacity ratio

5 Conclusion

Most of the developments witnessed in recent years in the area of exact algorithms for the SVRP have been centered on the application of the Integer L-Shaped Method proposed by Laporte and Louveaux in 1993. As the optimality cut (7) central to this method can be sometimes quite weak, computational enhancements are necessary to make this method more effective. As suggested by Laporte and Louveaux (1993), the cut can be lifted by examining neighbors of the current solutions. In addition, lower bounding functionals on the value of the expected recourse $Q(x)$ can sometimes be derived, as exemplified by the work of Hjorring and Holt (1996). It is encouraging to note that several medium size SVRPs can now be solved to optimality while this was impossible only a few years ago.

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References

[1] J.F. Benders. Partitioning procedures for solving mixed-variables programming problems. *Numerische Mathematik*, 4:238–252, 1962.

[2] D.J. Bertsimas. A vehicle routing problem with stochastic demand. *Operations Research*, 40:574–585, 1992.

[3] D.J. Bertsimas, P. Jaillet, and A.R. Odoni. A priori optimization. *Operations Research*, 38:1019–1033, 1990.

[4] J.R. Birge, and F.V. Louveaux. *An Introduction to Stochastic Programming*. Springer-Verlag, New York, 1997.

[5] G.B. Dantzig, D.R. Fulkerson, and S.M. Johnson. Solution of a large-scale traveling-salesman problem. *Operations Research*, 2:392–410, 1954.

[6] M. Dror, and P. Trudeau. Stochastic vehicle routing with modified savings algorithm. *European Journal of Operational Research*, 23:228–235, 1986.

[7] M.L. Fisher. Vehicle routing. In M.O. Ball, T.L. Magnanti, C.L. Monma and G.L. Nemhauser, editors, *Network routing*, volume 8 of *Handbooks on Operations Research and Management Science*, pages 1–33. North-Holland, Amsterdam, 1995.

[8] M. Gendreau, G. Laporte, and R. Séguin. An exact algorithm for the vehicle routing problem with stochastic customers and demands. *Transportation Science*, 29:143–155, 1995.

[9] M. Gendreau, G. Laporte, and R. Séguin. Stochastic vehicle routing. *European Journal of Operational Research*, 88:3–12, 1996.

[10] C. Hjorring, and J. Holt. *New optimality cuts for a single-vehicle stochastic routing problem*, 1996. Unpublished manuscript.

[11] P. Jaillet. A priori solution of a traveling salesman problem in which a random subset of customers are visited. *Operations Research*, 36:929–936, 1988.

[12] G. Laporte. Vehicle routing. In M. Dell' Amico, F. Maffioli, and S. Martello, editors, *Annotated Bibliographies in Combinatorial Optimization*, pages 223–240, Wiley, Chichester, 1997.

[13] G. Laporte, and F.V. Louveaux. The integer L-shape method for stochastic integer problems with complete recourse. *Operations Research Letters* 13:133–142, 1993.

[14] G. Laporte, F.V. Louveaux, and H. Mercure. Models and exact solutions for a class of stochastic location-routing problems. *European Journal of Operational Research*, 39:71–78, 1989.

[15] G. Laporte, F.V. Louveaux, and H. Mercure. The vehicle routing problem with stochastic travel times. *Transportation Science* 26:161–170, 1992.

[16] G. Laporte, F.V. Louveaux, and H. Mercure. A priori optimization of the probabilistic traveling salesman problem. *Operations Research* 42:543–549, 1994.

[17] G. Laporte, Y. Nobert, and M. Desrochers. Optimal routing under capacity and distance restrictions. *Operations Research* 33, 1050–1073, 1985.

[18] M.W. Padberg, and M. Grötschel. Polyhedral computations. In E.L. Lawler, J.K. Lenstra, A.H.G. Rinnooy Kan, and D.B. Shmoys, editors, *The traveling salesman problem. A guided tour of combinatorial optimization*, Wiley, Chichester, pages 307–360, 1985.

[19] R. Séguin. Problèmes stochastiques de tournées de véhicules. Ph.D. Thesis, Département d'informatique et de recherche opérationnelle, Publication CRT–979, Centre de recherche sur les transports, Université de Montréal.

[20] W.R. Stewart Jr., and B.L. Golden. Stochastic vehicle routing: A comprehensive approach. *European Journal of Operational Research* 14:371–385, 1983.

[21] R. Van Slyke, and R.J.-B. Wets. L-shaped linear programs with applications to optimal control and stochastic programming. *SIAM Journal of Applied Mathematics* 17:638–663, 1969.